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Separating Solution of a Quadratic Recurrent Equation

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Abstract In this paper we consider the recurrent equation

$$\Lambda_{p+1} = \frac{1}{p} \sum_{q=1}^{p} f\left(\frac{q}{p+1}\right) \Lambda_q \Lambda_{p+1-q}$$

for $p \geq 1$ with $f \in C[0, 1]$ and $\Lambda_1 = y > 0$ given. We give conditions on f that guarantee the existence of $y^{(0)}$ such that the sequence Λ_p with $\Lambda_1 = y^{(0)}$ tends to a finite positive limit as $p \to \infty$.

Keywords Separating solution · Quadratic recurrent equation · Mellin transform

1 Introduction

The following problem arose in the joint papers of the first author and Dong Li (see [2, 3]). Let f be a continuous real-valued function on [0, 1]. Define the sequence Λ_p for $p = 1, 2, \ldots$ by

$$\Lambda_{p+1} = \frac{1}{p} \sum_{q=1}^{p} f\left(\frac{q}{p+1}\right) \Lambda_q \Lambda_{p+1-q} \tag{1}$$

To J. Froehlich and T. Spencer with love and admiration.

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and set $\Lambda_1 = y \geq 0$. We shall occasionally write $\Lambda_p(y)$ to emphasize the dependence of Λ_p on the initial value y. It is clear that $\Lambda_p(cy) = c^p \Lambda_p(y)$. Therefore if $\Lambda_p(y) \to \infty$ as $p \to \infty$ and $c > 1$, then $\Lambda_p(y') \to \infty$ as $p \to \infty$ where $y' = cy$. On the other hand if $\Lambda_p(y) \to 0$ and $0 < c < 1$, then $\Lambda_p(y') \to 0$. Thus $\Lambda_p(y) \to \infty$ for $y \in (y^+, \infty)$, and $\Lambda_p(y) \to 0$ for $y \in (0, y^-)$, where $y^+ = \inf\{y\colon \Lambda_p(y) \to \infty\}$ and $y^- = \sup\{y\colon \Lambda_p(y) \to 0\}$. It is a natural question whether $y^+ = y^- = y^{(0)}$ and whether $\Lambda_p(y^{(0)}) \to \text{const}$ as $p \to \infty$. It is easy to see that this constant must be $(\int_0^1 f(x)dx)^{-1}$, and it is our first assumption that the last integral is positive. It is enough to consider the case $\int_0^1 f(x)dx = 1$ because if $\tilde{f}(x) = Kf(x)$ for a constant K, then $\tilde{\Lambda}_p(y) = K^{-1}\Lambda_p(y)$. If the answer to our question is affirmative then $\Lambda_p(y^{(0)})$ is called the separating solution of (1).

This problem was considered previously in [1, 4]. The analysis in [1] covered the case $f(x) = 6x^2 - 10x + 4$ needed in [2]. The analysis in [4] was based on a different idea but unfortunately had a number of gaps. This paper is a modified and corrected version of [4].

Before we give the assumptions we impose on f, we remark that $f(x)$ and $f(1-x)$ produce identical sequences. Therefore the existence of a separating solution depends only on $f_1(x) = f(x) + f(1-x)$. Of course establishing existence of a solution for f guarantees its existence for g if $g_1 = f_1$. Given $f_1(x)$ one can find $f(x)$ so that $f(1) = 0$. Thus we assume that $f(1) = 0$ without loss of generality. Now we impose the following conditions on f:

1. $f \in C^2[0, 1]$;
2. f_1 is positive on $[0, 1] \cap \mathbf{Q}$;
3. all complex $\sigma \neq 1$ satisfying $\int_0^1 t^\sigma f_1(t)dt = 1$ have the property that $\operatorname{Re}\sigma < 0$;
4. a numerical condition to be explained later.

Observe that an assumption similar to 2 is necessary as Λ_p will vanish for p sufficiently large if f_1 vanishes on too large a set (e.g., if $f_1(\frac{1}{2}) = 0$). Assumption 2 effectively ensures that $\Lambda_p > 0$ for all p. Finally we introduce functions

$$f_2(x) = -(xf(x))' \quad \text{and} \quad f_3(x) = -\frac{1}{x^2}\int_0^x tf_2(t)dt.$$

Define $a_p > 0$ for $p \geq 1$ by the condition $\Lambda_p(a_p) = 1$. Assumption 2 above makes this possible. The strategy of the proof will be to show that $a_p \to a_\infty$ sufficiently rapidly. Take positive constants A and B with $B < 1 < A$ and consider the inequalities

$$B \leq a_p, \qquad |a_p - a_{p-1}| \leq A/p^{2+\delta}, \tag{2}$$

where p is given and $\delta \in (0, \frac{1}{2})$ will be chosen later and will depend on f_1.

Theorem (Main Theorem) *Let f satisfy assumptions* 1–3 *above. If for some p_0 (depending on A, B, and f_1) the inequalities* (2) *hold for $p \leq p_0$, then they are valid for all $p \geq 1$.*

Our proof will be inductive. We shall assume (2) for $p \leq r$ and prove it for $p = r+1$. This will imply that the limit $\lim_{p\to\infty} a_p = a_\infty$ exists and $\Lambda_p(a_\infty)$ will be the desired separating solution.

The rest of the paper is structured as follows. In Sect. 2 we derive a recurrent equation for a_p. In Sect. 3 we solve this equation using the inductive hypothesis. The last section consists of numerical analysis and outlines further research on the problem.

2 Recurrent Equation for a_p

We shall denote absolute constants by C with superscripts in the course of this calculation. We have that

$$\Lambda_{p+1}(a_{p+1}) - \Lambda_{p+1}(a_p) = -(\Lambda_{p+1}(a_p) - \Lambda_p(a_p)). \tag{3}$$

Put $\gamma = \frac{p_1}{p}$, $p_2 = p - p_1$, $\gamma' = \frac{p_1}{p+1}$. Then

$$\begin{aligned}\Lambda_{p+1}(a_p) &= \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')\Lambda_{p_1}(a_p)\Lambda_{p_2+1}(a_p) = \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')(\Lambda_{p_1}(a_p)-1)(\Lambda_{p_2+1}(a_p)-1)\\ &\quad + \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')(\Lambda_{p_1}(a_p)-1) + \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')(\Lambda_{p_2+1}(a_p)-1) - \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')\\ &= \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma')(\Lambda_{p_1}(a_p)-1)(\Lambda_{p_2+1}(a_p)-1)\\ &\quad + \frac{1}{p}\sum_{p_1=1}^{p} f_1(\gamma')(\Lambda_{p_1}(a_p)-1) - \frac{1}{p}\sum_{p_1=1}^{p} f(\gamma').\end{aligned}$$

A similar formula can be written for $\Lambda_p(a_p)$:

$$\begin{aligned}\Lambda_p(a_p) &= \frac{1}{p-1}\sum_{p_1=1}^{p-1} f(\gamma)(\Lambda_{p_1}(a_p)-1)(\Lambda_{p_2}(a_p)-1)\\ &\quad + \frac{1}{p-1}\sum_{p_1=1}^{p-1} f_1(\gamma)(\Lambda_{p_1}(a_p)-1) - \frac{1}{p-1}\sum_{p_1=1}^{p-1} f(\gamma).\end{aligned}$$

Subtracting $\Lambda_p(a_p)$ from $\Lambda_{p+1}(a_p)$ we get

$$\begin{aligned}\Lambda_{p+1}(a_p) - \Lambda_p(a_p) &= \frac{1}{p} f\left(\frac{p}{p+1}\right)(\Lambda_p(a_p)-1)(\Lambda_1(a_p)-1)\\ &\quad + \sum_{p_1=1}^{p-1}\left(\frac{1}{p}f(\gamma') - \frac{1}{p-1}f(\gamma)\right)(\Lambda_{p_1}(a_p)-1)(\Lambda_{p_2}(a_p)-1)\\ &\quad + \frac{1}{p}\sum_{p_1=1}^{p-1} f(\gamma')(\Lambda_{p_1}(a_p)-1)(\Lambda_{p_2+1}(a_p)-\Lambda_{p_2}(a_p))\\ &\quad + \frac{1}{p} f_1\left(\frac{p}{p+1}\right)(\Lambda_{p-1}(a_p)-1)\\ &\quad + \sum_{p_1=1}^{p-1}\left(\frac{1}{p}f_1(\gamma') - \frac{1}{p-1}f_1(\gamma)\right)(\Lambda_{p_1}(a_p)-1)\\ &\quad + \frac{1}{p} f\left(\frac{p}{p+1}\right) - \sum_{p_1=1}^{p-1}\left(\frac{1}{p}f(\gamma') - \frac{1}{p-1}f(\gamma)\right) = \sum_{j=1}^{7} I_p^{(j)}.\end{aligned}$$

We estimate $I_p^{(j)}$. It will be shown that $I_p^{(5)}$ is the main term while the others have a smaller order of magnitude. This term produces the recurrent equation that we shall analyze in Sect. 3

It is readily seen that $I_p^{(4)} = \varepsilon_p^{(1)}$, where $|\varepsilon_p^{(1)}| \le \frac{C^{(1)}A}{Bp^{2+\delta}}$. The reasoning is as follows. Rewrite the term as

$$\frac{1}{p}\left(f_1(1) - f'(\xi)\frac{1}{p+1}\right)\left(\Lambda_{p-1}(a_{p-1})\left(\frac{a_p}{a_{p-1}}\right)^{p-1} - 1\right).$$

It is clear how to bound the second term in the first factor. The second factor can be written as

$$\sum_{k=1}^{p-1}\left(\frac{a_p - a_{p-1}}{a_{p-1}}\right)^k \binom{p-1}{k}$$

whence it is easy to see that it is bounded by const $\cdot \frac{A}{Bp^{1+\delta}}$. The estimate for the fourth term follows.

We go on to

$$I_p^{(5)} = \sum_{p_1=1}^{p-1}\left(\frac{1}{p}f_1(\gamma') - \frac{1}{p-1}f_1(\gamma)\right)(\Lambda_{p_1}(a_p) - 1).$$

For the first factor in the sum we get

$$\frac{f_2(\gamma')}{p(p-1)} + \varepsilon_p^{(2)}$$

where $|\varepsilon_p^{(2)}| \le \frac{C^{(2)}}{p^3}$. The second factor is more complicated and we first rewrite it as

$$\frac{a_p - a_{p_1}}{a_{p_1}}p_1 - \sum_{k=2}^{p_1}\left(\frac{a_p - a_{p_1}}{a_{p_1}}\right)^k \binom{p_1}{k}.$$

The last term of this expression is not more than $\frac{C^{(3)}}{p_1^{2\delta}}(\frac{A}{B})^2$. Multiplying out gives the following expression:

$$I_p^{(5)} = \sum_{p_1=1}^{p}\frac{\gamma f_2(\gamma')}{p-1}\frac{a_p - a_{p_1}}{a_{p_1}} + \varepsilon_p^{(3)}$$

and $|\varepsilon_p^{(3)}| \le C^{(4)}\frac{1}{p^{1+2\delta}}(\frac{A}{B})^2$.

Now we deal with three relatively simple terms. Let us begin with the seventh one:

$$\begin{aligned} I_p^{(7)} &= -\sum_{p_1=1}^{p-2}\left(\frac{1}{p}f(\gamma') - \frac{1}{p-1}f(\gamma)\right) \\ &= -\sum_{p_1=1}^{p-2}\left[\left(\frac{1}{p}f\left(\frac{p_1}{p+1}\right) - \frac{1}{p-1}f\left(\frac{p_1}{p+1}\right)\right) + \frac{1}{p-1}\left(f\left(\frac{p_1}{p+1}\right) - f\left(\frac{p_1}{p}\right)\right)\right] \end{aligned}$$

$$= \frac{1}{p(p-1)} \sum_{p_1=1}^{p-2} \left[f\left(\frac{p_1}{p+1}\right) + \frac{p_1}{(p+1)} f'\left(\frac{p_1}{p+1}\right)\right] + \varepsilon_p^{(4)}$$

$$= \frac{p+1}{p(p-1)} \int_0^1 [f(\gamma) + \gamma f'(\gamma)] d\gamma + \varepsilon_p^{(5)}.$$

Our assumption that $f(1) = 0$ implies that the last integral vanishes. Thus, $I_p^{(7)} = \varepsilon_p^{(5)}$ and $|\varepsilon_p^{(5)}| \le \frac{C^{(5)}}{p^2}$.

It is easy to see that $I_p^{(1)} = 0$ and $|I_p^{(6)}| \le \frac{C^{(6)}}{p^2}$.

To estimate $I_p^{(2)}$ we rewrite it as

$$I_p^{(2)} = \sum_{p_1=1}^{p} \left(-\frac{\gamma' f'(\gamma') + f(\gamma')}{p(p-1)} + \varepsilon_p^{(6)} \right) \left[\sum_{k=1}^{p_1} \left(\frac{a_p - a_{p_1}}{a_{p_1}} \right)^k \binom{p_1}{k} \right]$$
$$\times \left[\sum_{k=1}^{p_2} \left(\frac{a_p - a_{p_2}}{a_{p_2}} \right)^k \binom{p_2}{k} \right].$$

The terms in the brackets are bounded by $C^{(7)} \frac{A}{Bp_1^{\delta}}$ and $C^{(8)} \frac{A}{Bp_2^{\delta}}$. Thus the estimate for this term becomes $C^{(9)} \frac{A}{Bp^{1+2\delta}}$.

Finally for the third term we need to estimate

$$\Lambda_{p_2+1}(a_p) - \Lambda_{p_2}(a_p).$$

It is not difficult to see that $|\Lambda_{p_2+1}(a_p) - \Lambda_{p_2}(a_p)| \le \frac{C^{(10)}}{p_2^{1+\delta}} (\frac{A}{B})^3$. Combining this with the remaining factors gives the bound $C^{(11)} \frac{1}{p^{1+2\delta}} (\frac{A}{B})^5$ for $I_p^{(3)}$. We have used the fact that $f(x) \le C(1-x)$ in the last step.

Now we can put the seven terms together and see that

$$\Lambda_{p+1}(a_{p+1}) - \Lambda_{p+1}(a_p) = -\sum_{p_1=1}^{p} \frac{\gamma f_2(\gamma')}{p-1} \frac{a_p - a_{p_1}}{a_{p_1}} + \varepsilon_p^{(7)}$$

where $|\varepsilon_p^{(7)}| \le \frac{C^{(12)}}{p^{1+2\delta}} (\frac{A}{B})^5$. A simple calculation gives the recurrent equation

$$(p+1) \frac{a_{p+1} - a_p}{a_p} = -\sum_{p_1=1}^{p} \frac{\gamma f_2(\gamma')}{p-1} \frac{a_p - a_{p_1}}{a_{p_1}} + \varepsilon_p^{(8)} \tag{4}$$

with $|\varepsilon_p^{(8)}| \le \frac{C^{(13)}}{p^{1+2\delta}} (\frac{A}{B})^5$.

Our objective in this section is to derive a recurrent equation for $b_p = p^2 \frac{a_p - a_{p-1}}{a_{p-1}}$. Thus we rewrite (4) using b_p rather than a_p. We take a positive integer $Q \le p$ and get

$$b_{p+1} = -p \left[\sum_{p_1=1}^{Q} + \sum_{p_1=Q+1}^{p} \right] \frac{\gamma f_2(\gamma')}{p-1} \frac{a_p - a_{p_1}}{a_{p_1}} + \varepsilon_p^{(9)}.$$

Now the sum from 1 to Q gives a contribution bounded by $\frac{C^{(14)}}{p^2}\frac{A}{B}Q^{1-\delta}$. For the sum from $Q+1$ to p we observe that

$$\prod_{q=p_1+1}^{p}\left(1+\frac{b_q}{p^2}\right)-1=\frac{a_p-a_{p_1}}{a_{p_1}}.$$

The left hand side can be written as

$$\sum_{q=p_1+1}^{p}\frac{b_q}{q^2}+\varepsilon_{p_1}^{(10)}$$

with $|\varepsilon_{p_1}^{(10)}|\le\frac{C^{(15)}}{p_1^3}(\frac{A}{B})^2$ provided Q is chosen sufficiently large and independent of p. Using this fact we simplify our equation to

$$b_{p+1}=-p\sum_{p_1=1}^{p}\frac{\gamma f_2(\gamma')}{p-1}\sum_{q=p_1+1}^{p}\frac{b_q}{q^2}+\varepsilon_p^{(11)}, \tag{5}$$

with $|\varepsilon_p^{(11)}|\le\frac{C^{(16)}}{p^{2\delta}}(\frac{A}{B})^5$. After changing the order of summation we obtain the equation

$$b_{p+1}=\frac{1}{p}\sum_{q=2}^{p}b_q f_3\left(\frac{q}{p}\right)+\varepsilon_p^{(12)} \tag{6}$$

with $|\varepsilon_p^{(12)}|\le\frac{C^{(17)}}{p^{2\delta}}(\frac{A}{B})^5$. This is the equation we set out to solve; it is effectively a linearized version of the original equation.

3 Analysis of the Recurrent Equation

It will be more advantageous to have a continuous equation rather than a discrete one. To this effect we need to define $b(x)$ that would agree with b_p when $x=p$. First we observe that (6) can be written as

$$b_p=\frac{1}{p}\sum_{q=2}^{p}b_q f_3\left(\frac{q}{p}\right)+\varepsilon_p^{(13)}$$

with a different constant in the estimate for the error term. Now we can extend b as follows: set $b(x)=b_{\lfloor x\rfloor}$ with $b(x)=0$ on $[0,1)$. Then we have

$$\int_0^1 b(py)f_3(y)dy=\frac{1}{p}\sum_{q=2}^{p}b_q f_3\left(\frac{q'}{p}\right)$$

with $q'\in[q,q+1]$. It is easy to see that this sum differs from the one in the recurrent equation by not more than $\frac{C}{p^2}\sum_{q=1}^{p}b_q$. The new error term $\varepsilon(x)$ will incorporate this term as well as $\varepsilon^{(13)}$. It is also clear that we need to add lower order corrections to $\varepsilon(x)$ to ensure that $b(x)$ remains constant for non-integral x.

The equation to solve is now

$$b(x) = \int_0^1 b(xy) f_3(y)dy + \varepsilon(x). \tag{7}$$

The error term is $|\varepsilon(x)| \le \frac{C^{(18)}}{x^{2\delta}}$ (we are dropping the dependence on A and B for now).

Proposition 1 *Let f_3 be given as before and let*

$$\Sigma = \left\{\sigma \in \mathbf{C}\colon \int_0^1 t^\sigma f_3(t)dt = 1\right\}.$$

Then all $b(x)$ satisfying (7) with $\varepsilon(x)$ as above are (possibly infinite) linear combinations of elements of $\bigcup_{\sigma\in\Sigma}\{x^\sigma, x^\sigma \log x, \dots, x^\sigma \log^{k-1} x\} \cup \{b_\varepsilon(x)\}$ where $k = k(\sigma)$ denotes the multiplicity of σ, and the special solution $b_\varepsilon(x)$ has the property $|b_\varepsilon(x)| \le \frac{C^{(19)}}{x^{2\delta}}$.

Proof This proof can be carried out in a simpler way using the Mellin Transform, but we shall stick to the Fourier Transform as it is more common. To this end we set $x = e^\xi$, $y = e^{-\eta}$, $B(\xi) = b(e^\xi)$, $F(\eta) = -f_3(e^{-\eta})e^{-\eta}$, $E(\xi) = \varepsilon(e^\xi)$. We also extend f_3 to be zero on $(1, \infty)$. We get

$$B(\xi) = \int_{-\infty}^{\infty} B(\xi - \eta)F(\eta)d\eta + E(\xi).$$

Taking Fourier Transform of this equation yields

$$\hat{B}(\alpha) = \frac{\hat{E}(\alpha)}{1 - \hat{F}(\alpha)}.$$

Of course we only require that these are equal as distributions. Now $\hat{F}(-i\alpha) = \int_0^1 t^\alpha f_3(t)dt$, so we need to look where it attains the value one. It is precisely on the set $i\Sigma$. To invert $\hat{B}$, we shall integrate along a contour that goes around points in $i\Sigma$ (one can easily see them to be isolated) and stays on the real line otherwise. The integral away from the poles will give $b_\varepsilon(x)$ and can be bounded as follows. We know that $|E(\xi)| \le Ce^{-2\delta\xi}$ (hence the Fourier Transform is analytic in a strip centered at the real axis) and that $\frac{1}{1-\hat{F}(\alpha)}$ is meromorphic. Thus the decay rate for $(\frac{\hat{E}(\alpha)}{1-\hat{F}(\alpha)})^\vee(\xi) = b_\varepsilon(e^\xi)$ is the same as that for $E(x)$. Integrals near poles evaluate to residues at those poles, up to constants. For a simple pole at α' the residue is $e^{i\xi\alpha'}$. Residues at higher order poles are obtained in the same way. The result is immediate once we return to the original variables. □

The next proposition will allow us to better understand the structure of Σ.

Proposition 2 *With notation as above, the set*

$$\Sigma \cap \{\sigma \in \mathbf{C}\colon \operatorname{Re}\sigma > \sigma_0\}$$

is finite for each $\sigma_0 > -1$.

Proof Let us look only at the real part; in this calculation $\sigma = \mu + i\nu$. We have

$$\int_0^1 \cos(\nu \log t) t^\mu f_3(t) dt = -\frac{1}{\nu} \int_0^1 \sin(\nu \log t) \frac{d}{dt}(f_3(t) t^{\mu+1}) dt.$$

It is clear that the last expression tends to zero uniformly in μ as $\nu \to \infty$ provided $\mu > \sigma_0 > -1$. □

This Proposition allows us to study Σ more carefully. Since

$$f_3(t) = f_1(t) - \frac{1}{t^2} \int_0^t x f_1(x) dx \quad \text{and} \quad \int_0^1 f_1(t) dt = 2,$$

we always have $0 \in \Sigma$. Set

$$F_1(\sigma) = \int_0^1 t^\sigma f_1(t) dt \quad \text{and} \quad F_3(\sigma) = \int_0^1 t^\sigma f_3(t) dt.$$

Then

$$\frac{\sigma}{\sigma - 1} F_1(\sigma) - \frac{1}{\sigma - 1} = F_3(\sigma)$$

for $\sigma \neq 1$. This means that it suffices to look for solutions to $F_1(\sigma) = 1$. It is easy to see that $F_1(\sigma) < 1$ when $\operatorname{Re}\sigma > 1$ even without Assumption 3. It is also clear that $F_3(1) \neq 1$. Therefore Assumption 3 effectively says that there are no solutions to $F_3(\sigma) = 1$ in the strip $0 \leq \operatorname{Re}\sigma \leq 1$ with the exception of $\sigma = 0$. Thus $\sigma = 0$ is the solution with the largest real part. However, this solution is extraneous to our problem because it implies that $\frac{a_p - a_{p-1}}{a_{p-1}} \sim C/p^2$ and thus

$$\left(1 - \frac{a_\infty - a_p}{a_p}\right)^p \to e^{-C}.$$

This is only possible when $C = 0$, so this solution does not work in our situation. To this end we define $\Sigma' = \Sigma \setminus \{0\}$.

Suppose Σ' is nonempty and let $\sigma_1 = \max\{\operatorname{Re}\sigma : \sigma \in \Sigma'\} < 0$; it exists by Proposition 1. Then choose δ so that $\sigma_1 < -\delta < 0$. Then the slowest decaying solution b_p behaves at worst like p^{σ_1} and

$$\left(1 - \frac{a_\infty - a_p}{a_p}\right)^p \to 1.$$

This means that a_∞ is the desired separating solution. If Σ' is empty, define $\delta = \frac{1}{4}$ and $\sigma_1 = -\frac{1}{2}$. Then the same result holds. It is clear that A and B remain bounded in either case.

4 Numerical Analysis

While the Main Theorem is true for f satisfying Assumption 3, numerical calculations suggest that it is true even without this assumption. First we show how asymptotics work for a function satisfying this assumption, say $f(x) = 6x^2 - 10x + 4$. From Fig. 2 we see that $pb_p \to 0$, which is expected as $\Sigma' = \emptyset$ for this function. It is reasonable to infer from Fig. 1 that a separating solution exists and that the starting value is approximately 1.412729.

Fig. 1 The sequence a_p for $f(x) = 6x^2 - 10x + 4$

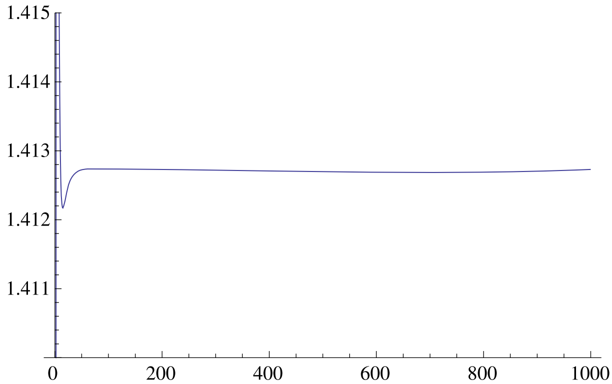


Fig. 2 The sequence pb_p for $f(x) = 6x^2 - 10x + 4$

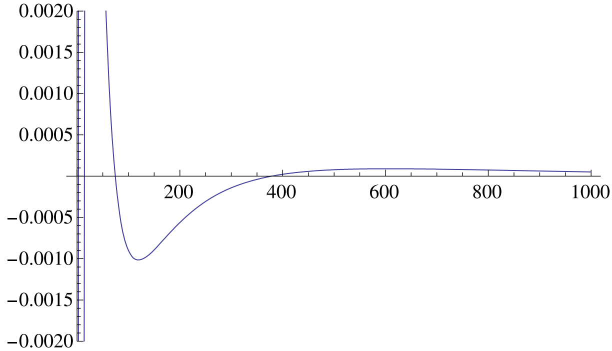
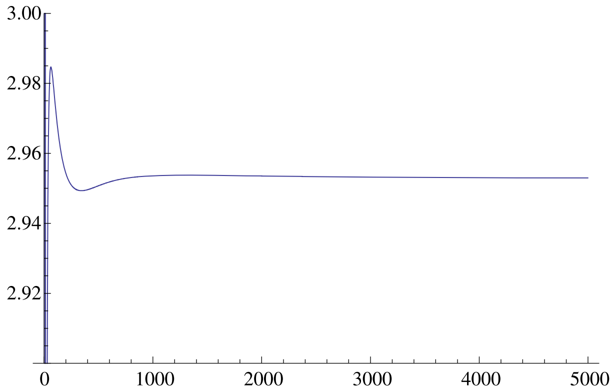


Fig. 3 The sequence a_p for $f(x) = 9x^8$



Next we consider the function $f(x) = 9x^8$. It has $\Sigma' \approx \{-0.234067 \pm 2.11581i\}$. Figure 4 shows $p^{-\sigma_1} b_p$ and $\cos(\operatorname{Im}\sigma \log p)$ (the smaller graph is the cosine). We see that the consecutive extrema of the rescaled b_p are at about the same absolute heights. In addition, we note that zeros of the two functions alternate. Therefore, it is plausible that $S\cos(\operatorname{Im}\sigma \log p + T)$ will coincide with our function for sufficiently large p. Figure 3 suggests that a separating solution exists and that $a_\infty \approx 2.95072$.

Finally we look at $f(x) = 13x^{12}$. It has $\Sigma' \approx \{0.105896 \pm 1.97567i\}$, and our theorem does not apply in this situation. Nevertheless numerics show that a_∞ exists, and its value is

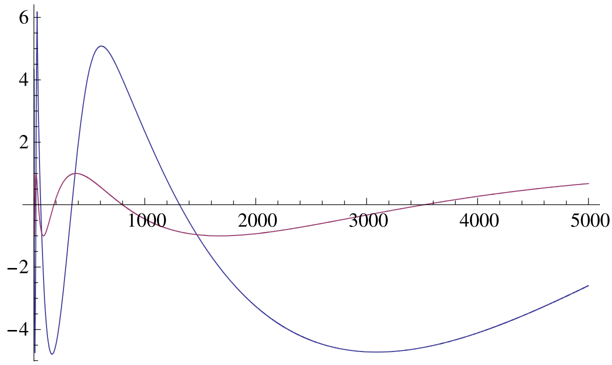


Fig. 4 The sequences $p^{0.234067} b_p$ and $\cos(2.11581 \log p)$ for $f(x) = 9x^8$

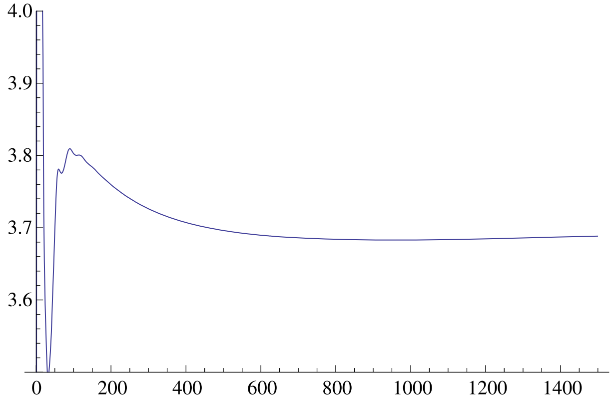


Fig. 5 The sequence a_p for $f(x) = 13x^{12}$

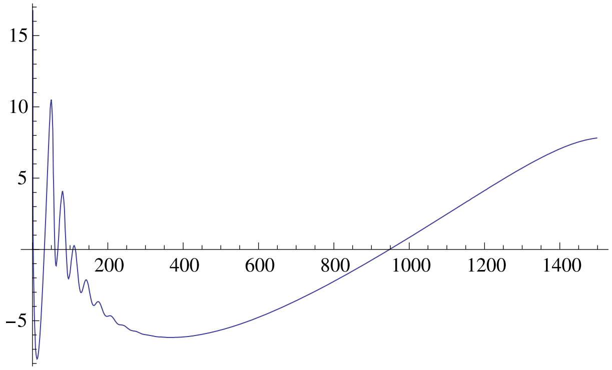


Fig. 6 The sequence b_p for $f(x) = 13x^{12}$

approximately 3.688371 (see Fig. 5). The sequence b_p in Fig. 6 doesn't seem to follow the asymptotic prescribed by σ_1. It is unclear how to pick δ for such a function since the error term $\varepsilon_p^{(12)}$ does not have good decay when $\delta < 0$. Apparently Assumption 3 is not necessary for the Main Theorem to hold, but in this case the structure of solutions to (6) is unclear.

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